Presentations for the Fundamental Group of a Homology 3-Spheres of Genus Two (多様体に於ける低次元トポロジーの問題)

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数理解析研究所講究録
309巻 1977年 72-79

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1. Introduction. The fundamental group $\Pi_1(M)$ of a 3-manifold M of genus two has two defining relations with two generators. Then the two defining relations are reduced to the canonical form (see Theorem 2). The canonical form is applied to determine whether homology 3-spheres are also homotopy 3-spheres in some simple cases, or not (Corollary 1, 2).

All spaces and maps considered here are polyhedral. S^n is a n-sphere and D^n is a n-disk. Let $M \subsetneq W$ be manifolds; the interior and boundary of M are denoted int(M), ∂M, respectively; M is properly embedded in W if $M \cap \partial W = \partial M$; N(M,W) is a regular neighborhood of M in W.

2. Presentations of $\Pi_1(M)$.

Let M be a closed orientable 3-manifold with a Heegaard splitting $(W_1, W_2; h)$ of genus two. Then the manifold M is the identification space $W_1 \cup_h W_2$ by the homeomorphism $h: \partial W_2 \longrightarrow \partial W_1$ of boundaries of solid tori W_1, W_2 of genus two. Let $\{D_{i1}, D_{i2}\}$ be a meridian disk pair of the solid torus $W_i (i=1,2)$, that is, D_{ij} (j=1,2) is a properly embedded 2-disk in W_i such that $W_i - (D_{i1} \cup D_{i2})$ is connected, and then $\{\partial D_{i1}, \partial D_{i2}\}$ is called a meridiam pair of W_i. Then we have;

Lemma 1. The manifold M is uniquely determined up to homeomorphism by circles $h(\partial D_{21})$, $h(\partial D_{22})$ (or $h^{-1}(\partial D_{11})$, $h^{-1}(\partial D_{12})$).

Proof. By the definition of $\{D_{21}, D_{22}\}$, the closure of $W_2 - (N(D_{21}, W_2) \cup N(D_{22}, W_2))$ is a 3-cell and so the lemma is valid.

Let $\Pi_1(W_i)$ be the fundamental group of the solid torus W_i and then

$\Pi_1(W_i)$ is a free group with two canonical generators a_i, b_i associated with $\{D_{i1}, D_{i2}\}$ such that a_i (or b_i) is a homotopy class generated by an oriented circle, which is disjoint from D_{i2} (or D_{i1}) and transversely intersects D_{i1} (or D_{i2}) at only one point, in W_i.

Let C be a oriented circle on ∂W_i, q a fixed point in C disjoint from $\partial D_{i1} \cup \partial D_{i2}$, and [C] the homotopy class in $\Pi_1(W_i)$ induced from the circle C. Furthermore let $\{ p_i \}_{i=1}^n$ be a sequence of points in C such that $\{ p_i \}_{i=1}^n = (\partial D_{i1} \cup \partial D_{i2}) \cap C$, p_1 is the first intersection next after the point q along the orientation of C, and the sequencial order follows from the orientation. Let ε_k be the intersection number between C and oriented circles ∂D_{i1}, ∂D_{i2} at the point p_k. Then $[C] = \bar{a}_1^{\varepsilon_1} \cdots \bar{a}_n^{\varepsilon_n}$ where $\bar{a}_k = a_i$ if $p_k \in C \cap \partial D_{i1}$, or $\bar{a}_k = b_i$ if $p_k \in C \cap \partial D_{i2}$. Let $L([C]) = n$ and $L([C])$ is called the length of the homotopy class [C] (or the circle C).

Let v_k (or w_k) be a homotopy class induced from the oriented circle $h(\partial D_{2k})$ (or $h^{-1}(\partial D_{1k})$) (k=1,2) in $\Pi_1(W_1)$ (or $\Pi_1(W_2)$). Then we have;

$$\begin{aligned} \Pi_1(M) &= \{a_1, b_1; v_1(a_1,b_1) = v_2(a_1,b_1) = 1\} \\ &= \{a_2, b_2; w_1(a_2,b_2) = w_2(a_2,b_2) = 1\} \end{aligned}$$

where $v_k(a_1,b_1)$ (or $w_k(a_2,b_2)$) is represented by the sequencial presentation above defined.

Let w be a word generated by symbols a, a^{-1}, b, b^{-1}, that is, $w = a_1 \cdots a_n$ where a_i is a symbol among a, a^{-1}, b, b^{-1}. Let α be a oparator such that $\alpha(w) = \bar{w}$ and $\bar{w} = a_n \cdots a_2 \cdot a_1$. Then the word w is said to be symmetric iff $\alpha(w) = w$. Let β_r be a oparator such that $\beta_r(w) = w'$ and $w' = a_{r+1} \cdots a_n \cdot a_1 \cdot a_2 \cdots a_r$. Then the word w is said to be freely symmetric iff $\beta_r\alpha(w) = w$ for some integer r. Then we have;

Theorem 1. Let C be a circle on ∂W_1 (or ∂W_2) such that C is not homologous to zero in ∂W_1 (or ∂W_2). Then [C] has a presentation in the free group $\Pi_1(W_1) = \{a_1, b_1; \text{ free}\}$(or $\Pi_1(W_2) = \{a_2, b_2; \text{ free}\}$) such that

(1) [C] is freely symmetric and

(2) $[C] = w_0^{-1}\cdot w_1\cdot w_2\cdot w_0\cdot w_3\cdot w_4$,

where w_1, w_2, w_3, w_4 are symmetric.

Prrof. If $C \cap (\partial D_{11} \cup \partial D_{12}) = \phi$, then $[C] = 1$. Consequently we may assume that $C \cap (\partial D_{11} \cup \partial D_{12}) \neq \phi$ up to a isotopy in ∂W_1. We cut ∂W_1 along the circle C. As a result, we obtain a torus T^2 with two holes: X^+, X^{-1} (note that $T^2 = \partial W_1 - \text{int}(N(C,\partial W_1))$). Under this operation, the circles ∂D_{11}, ∂D_{12} are cut up, and they turn into a collection of segments joining the holes in the torus T^2. Let $S(C) = S^+ \cup S_1 \cup S^-$ where S^+(or S^-) is a subset of segmennts in S(C) such that a segment $s \in S^+$(or S^-) is the one which joins ∂X^+(or ∂X^{-1}) to ∂X^+(or ∂X^{-1}), and S_1 is a subset of segments, in S(C), which join ∂X^{+1} to ∂X^{-1}. Let $\bar{T}^2$ be a copy of T^2 and a, b two circles in $\bar{T}^2$ such that $\bar{T}^2 - (a \cup b)$ has two components . Then there is a homeomorphism f from T^2 to $\bar{T}^2$ such that each segments of $f(S_1)$ transversely intersects which of circles a, b at only one point and each segments of $f(S^{+1})$ and $f(S^{-1})$ is mutually disjoint from both of circles a, b (see Figure 1). The existance of such a homeomorphism is trivially from the property of self-homeomorphisms of the torus and so we omit the proof. Then there is a symmetry respect of $a \cup b$ in $\bar{T}^2$, that is, there is a self-homeomorphism g of $\bar{T}^2$ such that $gf(X^{\pm 1}) = f(X^{\mp 1})$, $g(a) = a$, $g(b) = b$, $gf(S^+) = f(S^-)$, and $gf(S(C)) = f(S(C))$. Hence the first part of Theorem 1 is true and the second part of Theorem 1 refer to Figure 1. The proof is complete.

Furthermore we have;

Theorem 2. The fundamental group $\Pi_1(M)$ of the manifold M has a following presentation $\{\bar{a}, \bar{b}; v_1(\bar{a},\bar{b}) = v_2(\bar{a},\bar{b}) = 1\}$ associated with a meridian disk pair $\{D_{i1},D_{i2}\}$ of $W_i (i=1,2)$; (1) $v_1(\bar{a},\bar{b})$ satisfies the conditions (1),

(2) in Theorem 1, and $v_2(\bar{a},\bar{b}) = (w_1 \cdot w_2)^m \cdot (w_0 \cdot w_3 \cdot \bar{w}_0 \cdot w_2)^n$ where $\bar{w}_0 = \alpha(w_0)$.

Proof. Let C in Theorem 1 be $h(\partial D_{21})$. Then the homotopy class $v_1(a_1,b_1)$ generated by the circle $h(\partial D_{21})$ satisfies the conditions (1), (2) and then the homotopy class generated by the circle $h(\partial D_{22})$ is represented as $a^m \cdot e^n$ (m,n; relatively prime integers), since $h(\partial D_{22})$ is a simple closed curve disjoint from $h(\partial D_{21})$, $\mathrm{mod}(v_1(a_1,b_1) = 1)$. But $a = w_1 \cdot w_2$ and $e = w_0 \cdot w_3 \cdot \bar{w}_0 \cdot w_2$. The proof is complete.

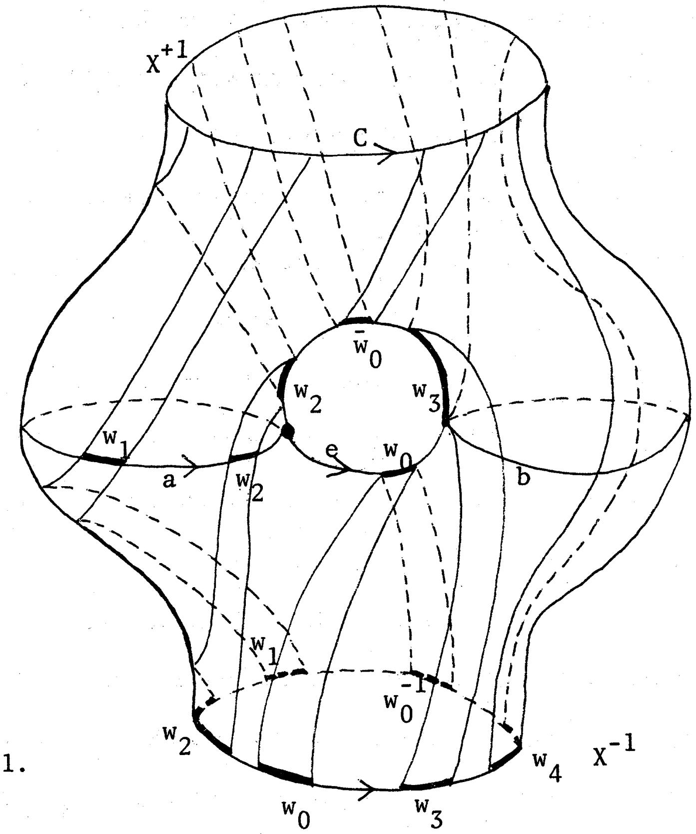

Figure 1.

Note that if m, n are relatively prime, the relation $(w_1 \cdot w_2)^m \cdot (w_0 \cdot w_3 \cdot \bar{w}_0 \cdot w_2)^n$ is realized as a circle C' in $\bar{T}^2$ and then the circle is deformed to the circle $h(\partial D_{22})$ $\mathrm{mod}(h(\partial D_{21}))$(m, n; appropriately fixed), that is, there is a sequence of circles $\{C_i\}_{i=1}^k$ such that $C_1 = h(\partial D_{22})$, $C_k = C'$, $(\bigcup_{i=1}^k C_i) \cap h(\partial D_{21}) = \phi$, and $C_i = C_{i-1} \# h(\partial D_{21})$ (#; a connected sum) along a arc τ_i which joins

$h(\partial D_{21})$ to C_{i-1} and is disjoint from $h(\partial D_{21}) \cup C_{i-1}$ except for the points $\partial \tau_i$, and consequently the splitting $\{h(\partial D_{21}), h(\partial D_{22})\}$ with $h(\partial D_{22}) = C_1$ is equivalent to the splitting $\{h(\partial D_{21}), h(\partial D_{22})\}$ with $h(\partial D_{22}) = C'$ (see [5]).

Corollary 1. If the circle $h(\partial D_{21})$ has $a_1^p \cdot b_1^q$ as the presentation with associated with $\{D_{11}, D_{12}\}$, all 3-manifolds given by splittings $\{h(\partial D_{21}), h(\partial D_{22})\}$ are not counterexamples of Poincaré conjecture.

Proof. By Theorem 2, the circle $h(\partial D_{22})$ has $(a_1^r \cdot b_1^t)^m \cdot b_1^{qn}$ as the presentation associated with $\{D_{11}, D_{12}\}$. Then we have that $\Pi_1(M) = \{a_1, b_1;\ a_1^p \cdot b_1^q = (a_1^r \cdot b_1^t)^m \cdot b_1^{qn} = 1$, p, q, r, t; positive integers with $r < p$, $t < q\}$. We may assume that 3-manifolds M are homology 3-spheres and so $P(a_1, b_1)$ (see [3]) is ± 1. Hence we have that $|(pt-qr)m + pqn| = 1$. Add the relation $a_1^p = b_1^q = 1$ to the group $\Pi_1(M)$ and let $A = a_1^r$, $B = b_1^t$. Let $G(p,q,m)$ be the group $\{A, B;\ A^p = B^q = (A \cdot B)^m = 1\}$. If $p, q, m \geqq 2$, then the gruop is non-trivial by Hempel [2]. Hence we may assume that $m = 1$. Then by the equation $|(pt-qr)m + pqr| = 1$, we have that $n = 0$. Consequently, the fundamental group $\Pi_1(M)$ is trivial only if $m = 1$, and $n = 0$. But we have the condition that $r < p$, $t < q$, and so the lemma is true.

Note that Corollary 1 and the extended result had been proved by Takahashi [4] using infinite cyclic covering spaces of tori with two holes.

Corollary 2. Let C be either of circles $h(\partial D_{21})$, $h(\partial D_{22})$. If $L(C) \leqq 10$, then 3-manifolds M given by Heegaard splittings $(W_1, W_2; h)$ are not counterexamples of Poincaré conjecture except for following Heegaard splittings;

(1) $b_1^3 = a_1^2 \cdot b_1 \cdot a_1 \cdot b_1 \cdot a_1^2$, $(b_1 \cdot a_1^3)^m \cdot (a_1 \cdot b_1 \cdot a_1^2 \cdot b_1 \cdot a_1^{-2})^n = 1$

(2) $b_1^3 = a_1 \cdot b_1 \cdot a_1^3 \cdot b_1 \cdot a_1$, $(b_1^2 \cdot a_1^{-1})^m \cdot (a_1 \cdot b_1 \cdot a_1^2 \cdot b_1)^n = 1$

(3) $a_1 = b_1^2 \cdot a_1^2 \cdot b_1 \cdot a_1^2 \cdot b_1^2$, $(b_1^2 \cdot a_1)^m \cdot (a_1^{-2} \cdot b_1 \cdot a_1^{-1})^n = 1$

(4) $b_1 = a_1 \cdot b_1^2 \cdot a_1^3 \cdot b_1^2 \cdot a_1$, $(a_1 \cdot b_1)^m \cdot (b_1 \cdot a_1^{-1} \cdot b_1^{-2} \cdot a_1^{-1} \cdot b_1^2)^n = 1$.

Proof. Let G(C) be a W-graph, of the circle C, associated with

$\{D_{11}, D_{12}\}$ of the solid torus W_1 (see [3]). We may assume that G(C) is type (1) with edge-parameters a, b, c, d (see Figure 2) by Theorem 1 in [3]. Let $N_1 = a + b + c$, $N_2 = a + b + d$ and then $L(C) = N_1 + N_2$. Suppose that 3-manifolds M are homology 3-spheres. Hence either of N_1, N_2 is an odd integer by Lemma 5 in [3] and by the symmetry of the graph G(C) we may assume that $N_1 \leqq N_2$. In the case that $L(C) < 10$, the corollary was proved by Takahashi [4]. Let $L(C) = 10$ and then following cases happen;

(1) $N_1 = 1$, $N_2 = 9$

(2) $N_1 = 3$, $N_2 = 7$

(3) $N_1 = 5$, $N_2 = 5$.

But the case(1) is trivial by Corollary 1 and in the case(2) there happen three cases; (2.1) $a = 1$, $b = 0$, $c = 2$, $d = 6$

(2.2) $a = 3$, $b = 0$, $c = 0$, $d = 4$

(2.3) $a = 2$, $b = 1$, $c = 0$, $d = 4$.

Then Case(2.1) is trivial by Corollary 1 and Case(2.2), Case(2.3) are reduced to the case with $L(C) = 7$, 9 respectively. Hence we check only the case(3). In Case(3), there happen following six cases;

(3.1) $a = 1$, $b = 0$, $c = 4$, $d = 4$

(3.2) $a = 3$, $b = 0$, $c = 2$, $d = 2$

(3.3) $a = 5$, $b = 0$, $c = o$, $d = 0$

(3.4) $a = 2$, $b = 1$, $c = 2$, $d = 2$

(3.5) $a = 4$, $b = 1$, $c = 0$, $d = 0$

(3.6) $a = 3$, $b = 2$, $c = 0$, $d = 0$

Then Case(3.1) is trivial by Corollary 1, in Case(3.2) and Case(3.3) any homology 3-spheres does not happen by Lemma 5 in [3], Case(3.5) and C(3.6) are reduced to the case with $L(C) = 7$, 9 respectively. The remained case is only Case(3.4). Under the condition that $a = 2$, $b = 1$, $c = 2$, $d = 2$, there are following circles;

(3.4.1) $b_1^3 = a_1^2 \cdot b_1 \cdot a_1 \cdot b_1 \cdot a_1^2$

(3.4.2) $b_1^3 = a_1 \cdot b_1 \cdot a_1^3 \cdot b_1 \cdot a_1$

(3.4.3) $a_1 = b_1^2 \cdot a_1^2 \cdot b_1 \cdot a_1^2 \cdot b_1^2$

(3.4.4) $b_1^2 \cdot a_1 \cdot b_1^2 = a_1^2 \cdot b_1 \cdot a_1^2$

(3.4.5) $a_1 \cdot b_1 \cdot a_1 = b_1^2 \cdot a_1^3 \cdot b_1^2$

(3.4.6) $b_1 = a_1 \cdot b_1^2 \cdot a_1^3 \cdot b_1^2 \cdot a_1$

(3.4.7) $a_1 \cdot b_1^3 \cdot a_1 = b_1 \cdot a_1^3 \cdot b_1$

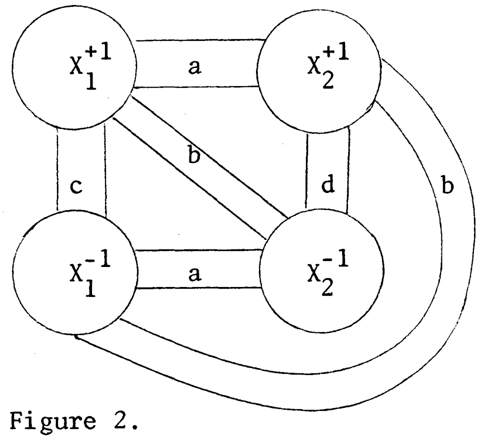

Figure 2.

Among them, in Case(3.4.4) any homology 3-spheres does not happen.

Case(3.4.5); By Theorem 2, we have that $\Pi_1(M) = \{a_1, b_1; a_1 \cdot b_1 \cdot a_1 = b_1^2 \cdot a_1^3 \cdot b_1^2, (a_1^{-1} \cdot b_1)^m \cdot (b_1 \cdot a_1 \cdot b_1^{-2} \cdot a_1 \cdot b_1^2)^n = 1$, m,n; relatively prime$\}$. Add the relation $a_1^4 = b_1^3 = 1$ to the group and let $G(m+n)$ be the group $\{A, B; A^4 = B^3 = (AB)^3 = (A^{-1}B)^{n+m} = 1\}$. To abelianize the group $\Pi_1(M)$, we obtain a 2×2 matrix $P(a_1,b_1)$;

$$P(a_1,b_1) = \begin{vmatrix} 1 & 3 \\ -m+2n & m+n \end{vmatrix}$$

The determinant of it is ± 1 by Lemma 5 in [3] and so the condition that $4m-5n = \pm 1$ holds. Hence $|n + m| \geq 2$ and $m \times n > 0$. If $|n + m| = 2$, then $m = n = 1$. But the length of the relation $(a_1^{-1} \cdot b_1) \cdot (b_1 \cdot a_1 \cdot b_1^{-2} \cdot a_1 \cdot b_1^2) = 1$ is exactly 9 and so the case with $|n + m| = 2$ is trivial by [4]. Suppose that $|n + m| > 2$. Let $G(n+m)$ be the group $\{A, B; A^4 = B^3 = (AB)^3 = (A^{-1}B)^{n+m} = 1, |n+m| > 2\}$ and the group $G(n+m)$ is non-trivial by [1].

Case(3.4.7); By Theorem 2, we have that $\Pi_1(M) = \{a_1, b_1; a_1 \cdot b_1^3 \cdot a_1 = b_1 \cdot a_1^3 \cdot b_1; (b_1^2 \cdot a_1)^m \cdot (b_1^{-1} \cdot a_1^{-1} \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot a_1)^n = 1$, m,n; relatively prime$\}$. Then the matrix $P(a_1,b_1)$; $\begin{vmatrix} 1 & -1 \\ m-n & 2m-n \end{vmatrix}$ is obtained and the determinant of the matrix is ± 1 and so $3m - 2n = \pm 1$.

Hence the condition that $2n + m \geqq 3$ holds. Add the relation $a_1^4 = b_1^4 = 1$ to the group $\Pi_1(M)$ and let $\bar{G}(2n+m)$ be the group $\{A, B; A^4 = B^4 = (AB^{-1})^3 = (B^2A)^{m+2n} = 1, |2n + m| \geqq 3\}$. Let $A = CB$ and $G(2n+m)$ the group $\{C, B; C^3 = B^4 = (CB)^4 = (CB^{-1})^{m+2n} = 1, |2n + m| \geqq 3\}$. Then the group $G(2n+m)$ is non-trivial by [1]. The proof of the corollary is complete.

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